Distributing The Difficulty Of The Problems In The Test

Team #863

Summary

During the investigation of the ways of the design an efficient test paper that could show or visualize the distribution of the students with varying ability, we discovered a model that is can be used for the designer of the test to have an excellent way of analyze the students' abilities

Our model was designed to show the how a group of people's abilities distributed in a chart. So the professor of the class can have better understanding about the class and also improve the quality of the teaching. By using the correct arrangement of the difficulty, the test paper can be more efficient for the understanding of the class. First, we considered how we should arrange the amount of the questions and the form of the questions. Should we use Q & A questions or the questions that only requires one to choose right or wrong? Should we give the test in infinite time that allows any students to have the chance to finish all the questions?

Secondly, how to quantize the abilities of the students? Do we use our measurement to measure them or do we use the highest score as the best?

The questions can be answered in the following passage.

1 Introduction and Problem Restatement

1.1 Introduction

Nowadays, the education system is gradually improving. However, what about the quality? The quality of the class is sometimes much more important than all the other things. By the word quality, we meant how much does the professor of the class know about his student (**How much knowledge do they know? How well can they apply the knowledge to question they met.**). By knowing above information, the professor can get clearer about how to arrange the class, how to pass on knowledge to the ones attend the class. If the quality of the test reduces, the result will lead to

On the other hand, the way how professor acquaints himself with the class is also another important problem.

The most common way of doing so is by using test papers, which the professor can use the score to analyze the distribution of the students ability. Therefore, how to properly distribute the difficulty of the problems on the test paper is a problem to solve. So as to, for the professor of the class, have the most accurate result of the distribution of the students in class with varying abilities.

Different methods were used by us in order to analyze various situations of the distribution of the class aptitude. The relationship between the efficiency of the test and the arrangement of the questions in the paper were especially considered.

1.2 Problem restatement

We tried to build a model which can help the professor to have a more effective way to arrange a test paper more efficiently. Firstly, we considered the ability of the student can be shown in figures. We adjusted the amount of the questions in the exam. Also, we addressed situations with a class that the ability of the student distributed in different forms of distribution (Normal distribution and the other forms of distribution).

The most important factor is the arrangement of the questions that can be used to visualize the students' abilities into data.

2 Definitions, Assumptions and Variables

2.1 Definition

1. Students' ability (quantized).
2. The arrangement of difficulty of each question of the test.
3. Students' concentration and vigor (However, the time it take to finish the test is small enough for any student to be fully concentrated in).
4. The time it takes for different students to finish a question.
5. The point for each part of the question.
6. All the questions are all Q & A question (For definition, see the first defined terms).

Now we define some terms what will be used in the whole paper.

Q & A Question: A question that assess many parts of the knowledge that also needs amounts of necessary steps to complete it.

Unit Solution: A step of the solution in any Q & A question that is indispensable and necessary.

Unit Points: The points that is given to the correct *unit solution* (Unit Points $=\frac{100\ (Full\ Score)}{n(A)}$, where n is the number of Q & A questions in the test. The unit points for every unit question are always the same.

Unit Time: The least time for a student to finish a Q & A question.

2.2 Assumptions

In order to streamline our model, we have made several key assumptions.

Assumption: Everyone who attended the class is on their best condition. Which means the test can exactly show the aptitude of the student.
Justification: Because the mental mood of the student is random, and is impossible to predict. Consider mental factors as a part in our model will make the difficulty of the building of the model beyond our ability.

Assumption: The time limit of the test is long enough for the top students to finish, and also small enough so that the students' vigor won't change.
Justification: Because the error caused by the tiredness is unpredictable. Also the better the student is the faster he can work out the problem, as well as higher

accuracy.

Assumption: Every question has a certain difficulty that if and only if the student have the corresponding ability to apply the knowledge to the question.
Justification: If the student can't get scores on the thing on they actually can do. OR if they can get scores on the things they don't even know about. The test will be invalid for indicating the students' ability.

Assumption: All the questions are required to show every step of the participants' solution.
Justification: If the question only ask for the final answer, it will be very hard for anyone to tell if the student really know about the knowledge or not.

Assumption: The time for a student to solve the problem follows a modified Gaussian function, which will be described later.
Justification: Students often behave in this way.

Assumption: Students will finish the questions one by one.
Justification: This is what the students normally do.

Assumption: The unit time is the same for every question.
Justification: Most questions can be decomposed into several Q & A questions which require the same time to finish.

Assumption: The professor has a prior knowledge about his/her students.
Justification: If not, the professor can't distribute the difficulty of problems properly.

Assumption: The quality of the test is measured by the linear correlation between the ability of the students and their marks.
Justification: This is a commonly used measure.

Assumption: The professor has a baseline about the students' knowledge.
Justification: For any students in university, students do need certain basic knowledge to get in. Otherwise it will be impossible for the professor to give the test.

2.3 Variables

T is the total time for a student to finish the test.
$\mathbf{T_n}$ is the total time for a student to finish the question **n**.
N is the number of unit questions on the test.
S is the score for a unit question. It is obvious that S=100/N.
$\mathbf{D_n}$ is the difficulty of the unit problem **n**.
$\boldsymbol{\sigma_n}$ is a number to show the time difference of students with different abilities to solve the problem. The test is modeled as a collection of the $\mathbf{D_n}$ and their respective $\boldsymbol{\sigma_n}$.
A is the quantified abilities of a student, satisfying 0<**A**<100.

3 Mathematical Model

3.1 Modeling a student doing a test

The process for a student to do a test is quite simple. The student tries to work out the first question, in time T1(measured in unit time),where

$$T1 = 4e^{(\frac{(A-D_1)}{\sigma_1})^2} + 1$$

If there is time left, he/she tries to finish the second question, and then the third, and so on. He/she gets points if and only if he finished the question before time **T** and his ability **A** is larger than the difficulty $\mathbf{D_n}$.
The procedure mentioned above can be written into pseudocode:

```
Input(D,σ,A,N)
Score ← 0
Time←0
n← 1
While Time<T And n<=N
         Time=Time+4*exp((A-Dn)/σn)^2)+1
If Dn < A
Score ← Score + Dn
 End If
   n←n+1
End While
Output(Score)
```

3.2 Estimating and optimizing the quality of the test

As mentioned before, the professor has a prior knowledge of his/her students. In order to estimate the quality of the test, one can generate some 'students' randomly whose ability follows the prior knowledge of the professor, and apply the test on them. By calculating the linear correlation between the ability of the students and their marks, one can know the quality of the test. Such modeling technique is called **Monte-Carlo simulating.** We used pattern search to optimize the quality, since pattern search doesn't require rapid evaluations of the function to be optimized.

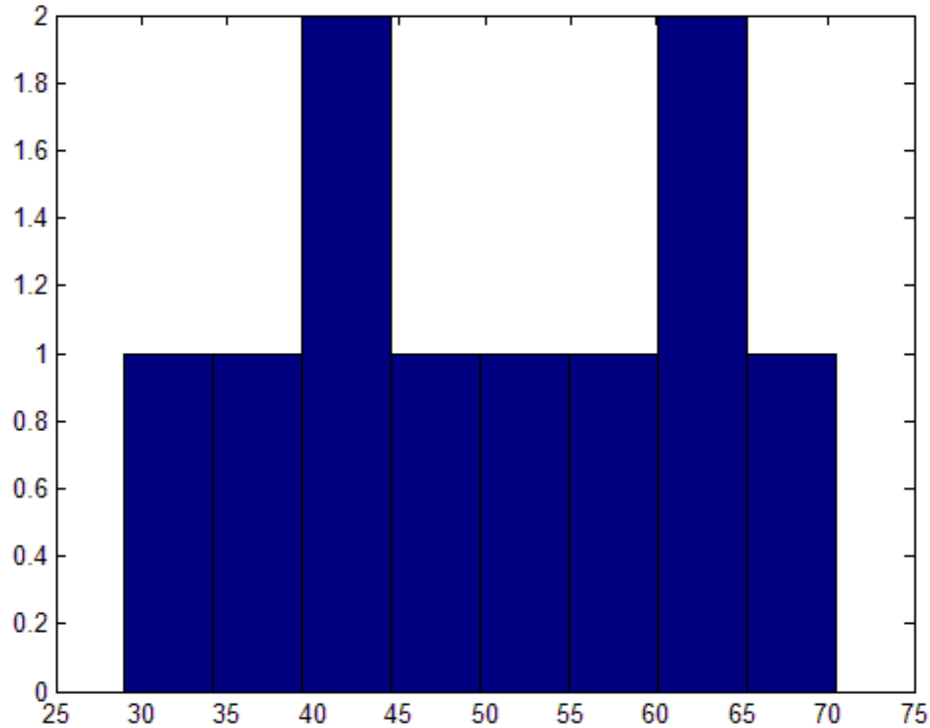
4 Solutions and explanation

4.1 The basic case: Normal distribution

We tested our model on a group of students whose abilities follows the normal distribution of mean 50 and standard deviation 10. The initial value used in pattern search is 10 problems of difficulty 50. The difficulties of the problems are shown below:

57.8112	39.5583	44.3895	48.9622	34.3018
28.9918	54.0287	62.3086	70.4942	65.2473

And the histogram:



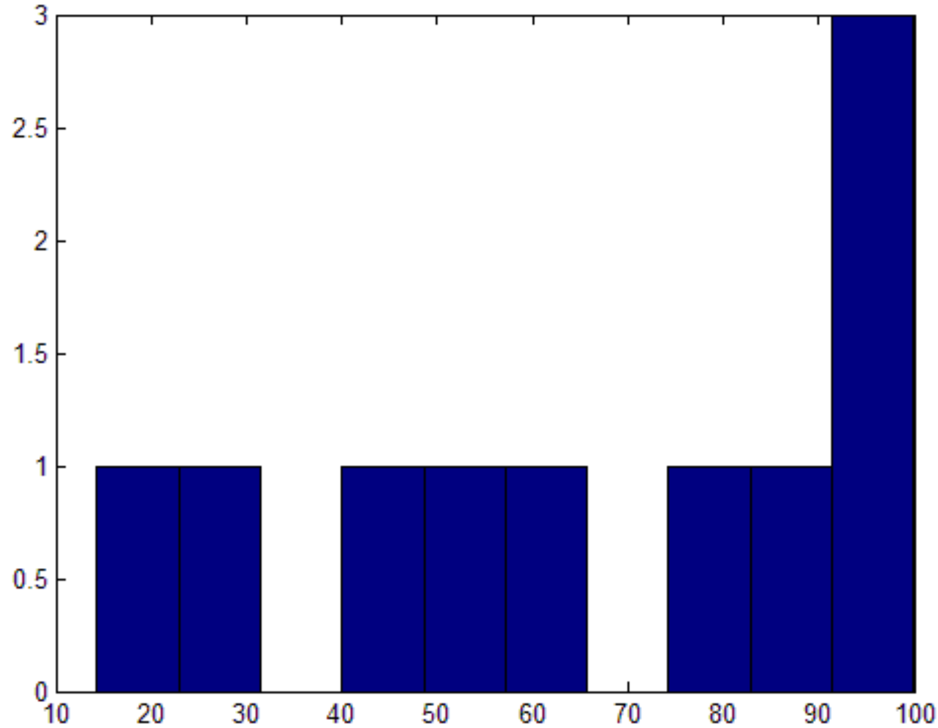
It appears that the difficulty of the test is distributed near difficulty 50,which is the mean ability of the students. Such a test will reflect students' ability more precisely.

4.2 The unusual case

What if the students' abilities follow an unusual distribution?By generating half of the students near 20 and half of the students near 80, what is get is this:

55.1275	76.3120	99.9862	59.9836	14.3151
40.9056	98.8119	99.0850	29.1046	88.4510

And the histogram:



As it is shown, the difficulties are near 20 and 80. The peak near 50 is caused by the outliers of the 20-students and the 80-students. Again, the test fits the students well.

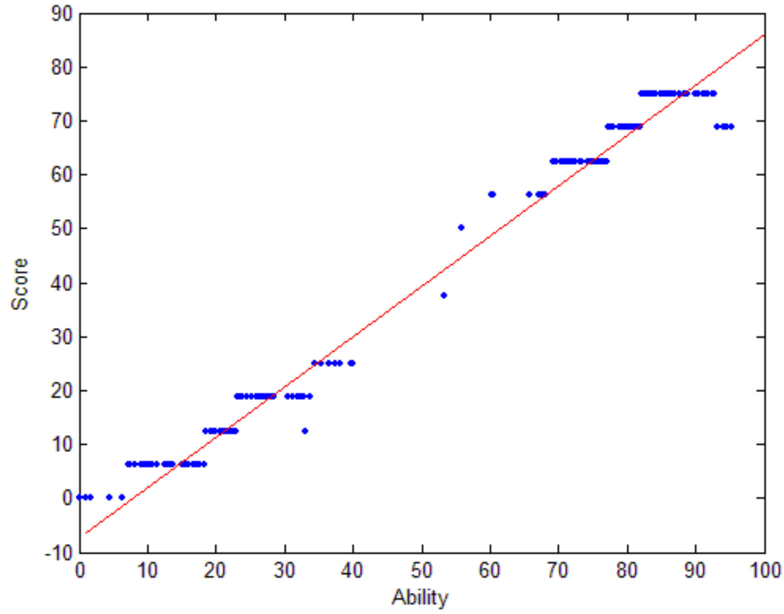
4.3 The 16-unit problem case

Because of simplicity, only the figures are presented:

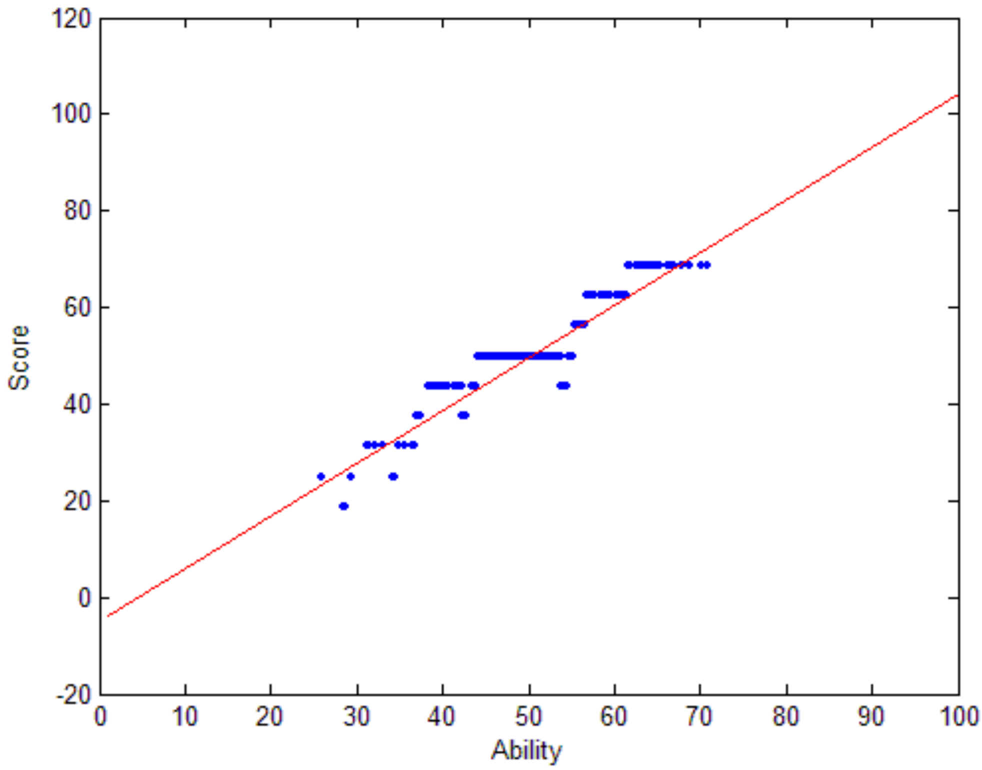
Difficulty for problem 4.2:

18.4409	46.9224	6.4781	98.8335
58.2792	42.3496	51.5512	33.3951
53.2907	22.5950	67.9807	76.0365
52.9823	64.0526	20.9069	37.9818

Correlation of ability to score (correlation coefficient is 0.994):

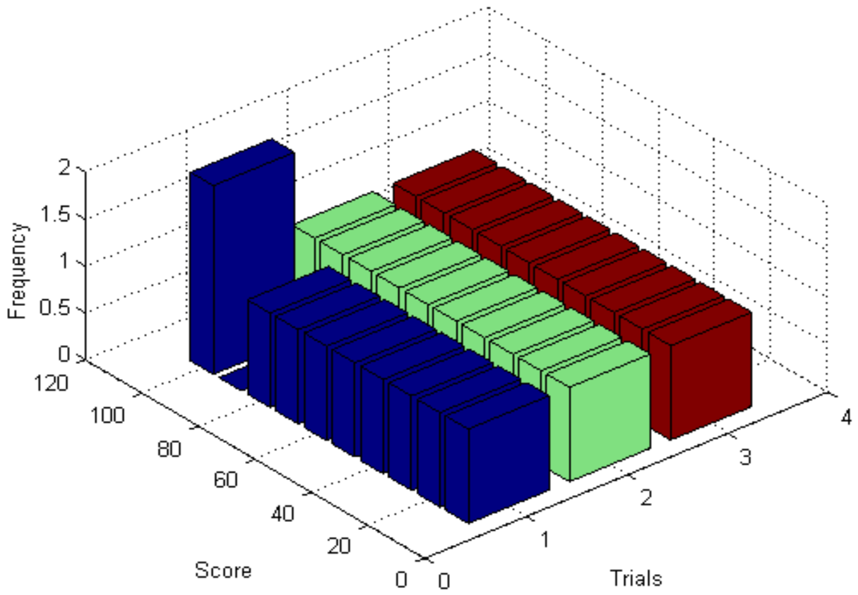


A strong linear correlation is presented above, as well as that in problem 4.1:



5 Sensitivity Analysis

As we all know, the little difference of the initial value of a optimizer will cause the result to vary a lot. By varying the initial values of Problem 4.1, it seems that the result varies only a little:



It is easy to deduce that our model is very stable and flexible.

6 Strength and Weakness

6.1 Advantages

a. Our model is very stable. For the reason that, the variables in the model are less than the initial one. Also 'Stable' means it is flexible for use, for any professor can use this model to test the aptitude for the student whatever the how the students' aptitude line chart present.

b. Because the ingenious simplicityin calculation. The statisticcan calculate and converge very fast which fits the high-pace-lifestyle nowadays.

c. Our model is build base on the reality. As a result it is consonant with our common sense, which meant this is functional in actual use.

d. The model use diagrams which 9s instant and easy to tell.

6.2 Disadvantages

a. This model is not exactly or completely solvable which means it can't give the answer directly or generate the diagram instantly. To solve this drawback the model requires more statistic and more data beforehand.

b. .This model is mainly based on a Gaussian assumption of the student's time on working out any problem.It's still unknown to us whether the assumption is showing the actual facts or not which lead to the possibility that the model maybe coarse.

c. The model we considered didn't take the psychological condition as one of the factor that may effects the accuracy of the student who will take the test. Also the accuracy of the model's prediction. Because the psychological condition of the student contains too much potentiality and randomization of it is unpredictable.

7. Conclusion

Our model was designed to show how a group of student's abilities is distributed in a class. So the professor of the class can have better understanding about the class and also improve the quality of the teaching. By using the correct arrangement of the difficulty, the test paper can be more efficient than traditional ones.

Positioning Minimum Satellites To Cover The Earth

Team #863

Summary

For the purpose of calculating the number of our positioning satellites and making a model of constellation, we created a visible model to describe the way in which satellites and the GPS receivers work together. With the help of Matlab, we programmed an animated-feature film to show the orbits around the earth and the areas where people can get accurate positions.

As we know, satellites move in different speeds according to their altitude. What's more, the angle between equatorial plane and a certain satellite's orbit determines the latitude in which it can cover. After inputting the two variables above, our program is able to calculate the covered area at a certain moment.

In order to measure the population of different regions of the world, we divide the surface of the earth into the densely populated area and the sparsely populated area. The results reflect in our program and are represented in a visual environment.

We tried enormous arrays of variables, varying the number of satellites and trying distinctive probably feasible constellations, until a highly consistent solution is worked out.

Our final result shows that the minimum value of N is 12.

1 Introduction and Problem Restatement

1.1 Introduction

Space, the final frontier of us, is the field that we are now exploring. Also, we take the advantages of the space to help us locate ourselves on planet earth. This is called the *Global Positioning System* (commonly known as GPS). GPS use satellites and certain ways of calculation to analyze and helped us to find our location in all most any place in this world.

The way how we arrange the satellite can also be a very important problem. The satellite we installed above the atmosphere should be functioning as we should let a great number of people benefits from the satellites we placed. Our model represents where to place the satellite, and how many we should have, so that, for most of the people on planet earth can receive the signal and get their instantaneous location

1.2 Problem restatement

The question of the placement of the satellite on the outer range of the earth can be a difficult question to solve. Our model mainly focuses on the way of the placement of the satellite of *Global Positioning System,* which can have fewer effects on the other satellites that has already be on the sky. Also have the best service for the people who use the device to get where he/she is.

First we considered how the satellite can be arranged on the orbit. Then it was the angles of the orbit were taken into consideration. Finally, it was how the most population of this planet distributed. In order to let 90% of the populaces on this world are able to evenly have the signal with ease and also locate themselves whenever (at least 95% of the time) they want.

The most important factor of the modeling is that the distribution and also the frequency of the rotation of the satellites on the decided orbit.

2 Definitions, Assumptions and Variables

2.1 Definition

1. The earth is a perfect sphere (r=6371km). The surface of the planet is smooth (without any mountains and other terrains)
2. The earth is separated into 3 segments (1. North of the 50°N 2. Between 50°N and 50°S 3. South of 50°S). The second segment contains 90% of the world population.
3. *Signal receivable* means that for any fixed point, the line segment that passes through it and the satellite doesn't intersect with the earth.
4. Accuracy means for anyone who use the GPS he is capable of receiving at least 4 signals from the satellite at any moment.
5. For any two satellites on the same orbit, the line which passes through it is also passing the centricity of the planet.
6. The constellation takes the centricity of the planet as the center of symmetry.
7. The orbits are at the height of **h** above sea level (2000km<**h**<36000km).

Now we define some terms what will be used in the whole paper.

Absolute Height: The distance between any item to the center of the earth.

2.2 Assumptions

In order to streamline our model, we have made several key assumptions.

Assumption: The thickness of atmosphere is even everywhere and do not deflect the signal.
Justification: In order to get a solvable model.

Assumption: We consider the satellite as a point.
Justification: Because its size can be omitted.

Assumption: To each satellite, the angle between its orbit and equatorial plane is a constant.
Justification: For saving variables.

Assumption: To each satellite, the normal of its orbit forms a distribution mentioned in 3.3.
Justification: For saving variables.

2.3 Variables

N is the total time for orbits.
θ is the dip angle for orbits.

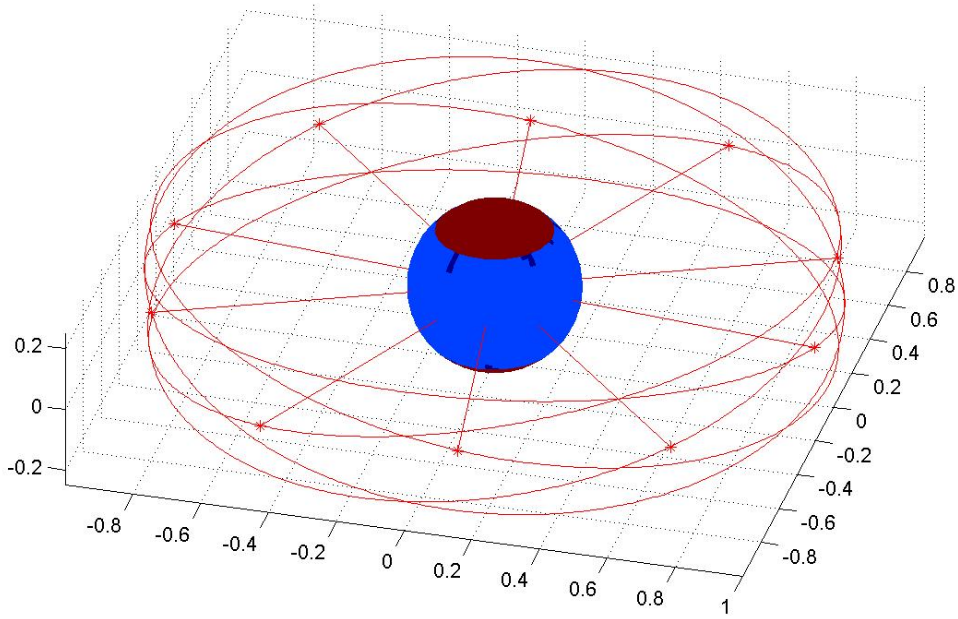
3 Mathematical Model

3.1 Modeling the motions of satellites

According to the original definition, the satellites move on a circular orbit. Referring to the international practice, the absolute height of the positioning satellite is usually 4 times than the absolute height of the Earth. A unit vector is used to represent the orbit, since there is only one circle which satisfied the constraints above with the vector as its normal. Besides the orbit, the phase difference of satellites is also a vital factor. We use uniform distribution in this paper.

3.2 Modeling the ground receivers

Since the exact calculation is too complex, we divide the surface of the earth into 4096 uniform regions. According to the original assumption, we consider the region to be covered only if the line segment that passes through it and the satellite doesn't intersect with the earth. For further more, the region is thought to get accurate positioning services if it's covered by over 4 satellites. We'll present the result in the following picture.



The part shown in light blue stands for the area that gets accurate GPS services. On the other hand, the part shown in dark blue stands for the area that is unable to get covered by enough satellites. The red part is the place which contains only 10% of the world population. The red dots are satellites circling around the earth. All the line segments shown in the picture point to the centricity of the planet.

We can get to know that the blue part has covered all the area from 50°S and 50°N, which is considered to be the densely populated area where over 90% of the world's population settles.

3.3 Modeling the number, directions and dip angles of the orbits

The number of the orbits is a vital variable to our model, for it is directly related with the number of satellites. According to our assumption, for each satellite there is a corresponding one on the opposite side of its orbit. We conducted several experiments using our program, and found that the model best meets our assumption when there was 5 orbits available. The 4-orbit case requires the satellites to be very far, which is impractical.

A unit vector is used to represent each orbit. In our paper, we used the model below:

1. Generate N equally-spaced points on the equator. Connect each point to the centre of the planet ,we get N vectors pointing from the center to a point on the equator.

2. Add z on each vector, where z is a vector which points to the North Pole.
3. Normalize each vector.

   Written into Cartesian coordinates:

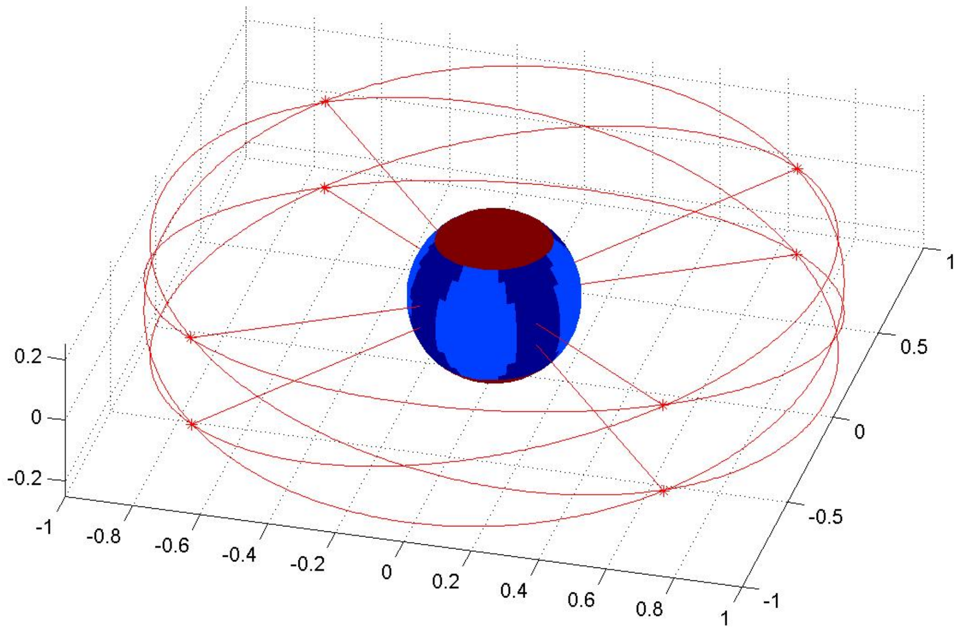
$$\overrightarrow{n_1} = \left(\sin\frac{360°\times i}{N}\cos\theta\,, \cos\frac{360°\times i}{N}cos\theta, \sin\theta\right) \qquad i = 1..N$$

All the satellites start from the point on the orbit where it passes through the equator plane. There are two such points for one orbit, which is the initial position for the two satellites on the orbit.

4 Solutions and explanation

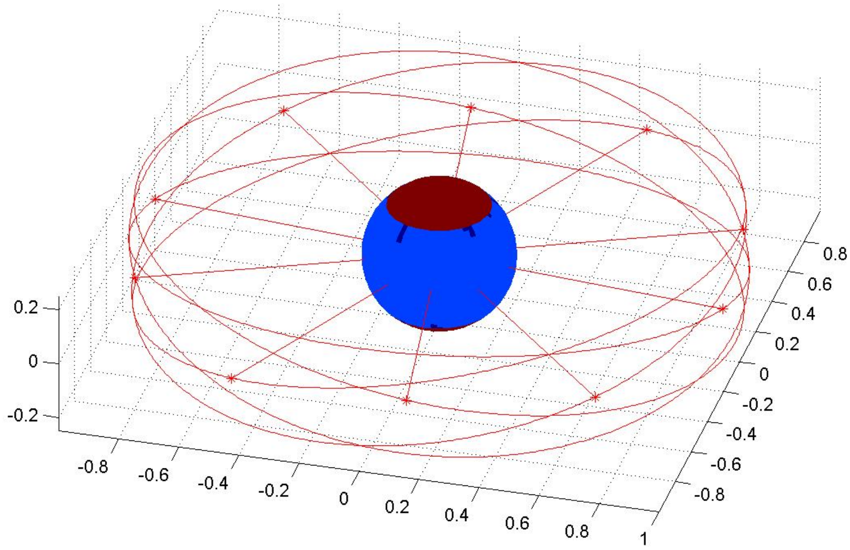
The 4-orbit, 5-orbit and 6-orbit case are presented below for comparison.

4.1 The 4-orbit case: Impractical



As it is shown, there are regions (shown in dark blue) which can't get positioning services. This region can be made very small, but it requires a extremely high orbit. Such an orbit is impractical for GPS devices.

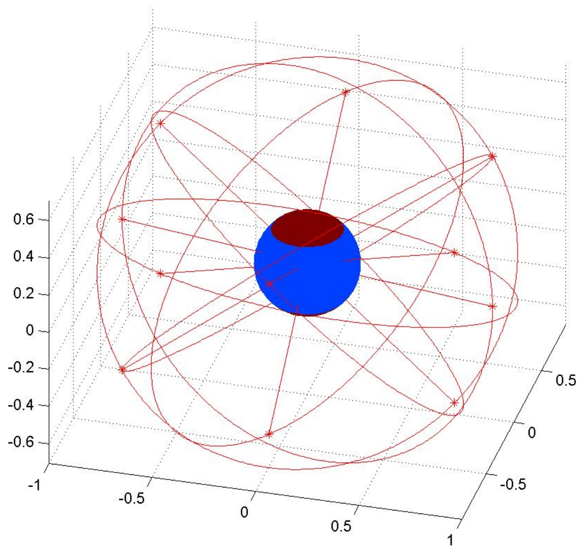
4.2 The 5-orbit case: Poor



The 5-orbit case is much better than the 4-orbit case. Numerical results show that the angle θ can't be bigger than 11 degrees. Such a small angle will yield poor positioning.

4.3 The 6-orbit case: Perfect

The 6-orbit case is perfect. The θ angle shown is 45 degrees, which yields perfect results. Therefore, there is no need to use more satellites than 12.



5 Strength and Weakness

5.1 Advantages

a. Our model uses only two variables, which increase the speed enormously. One can try the variables for many times.

b. Our model is simple, which enables the possibility to analyze it theorically.

5.2 Disadvantages

a. This model only explores a small space of the whole possible orbits.

b. The accuracy of the position signal is not calculated precisely.

6 Conclusion

Our model was designed to show how to position satellites in order to cover 90% of the position. Such a model can be used in many situations. By tuning the variables, better constellations can be obtained.